

# Thompson's conjecture for alternating group

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*Abstract:* Let  $G$  be a finite group, and let  $N(G)$  be the set of sizes of its conjugacy classes. We show that if a finite group  $G$  has trivial center and  $N(G)$  equals to  $N(Alt_n)$  or  $N(Sym_n)$  for  $n \geq 23$ , then  $G$  has a composition factor isomorphic to an alternating group  $Alt_k$  such that  $k \leq n$  and the half-interval  $(k, n]$  contains no primes. As a corollary, we prove the Thompson's conjecture for simple alternating groups.

*Key words:* finite group, alternating group, conjugacy class, Thompson's conjecture.

## 1 Introduction

Consider a finite group  $G$ . For  $g \in G$ , let  $g^G$  denote the conjugacy class of  $G$  containing  $g$ , and  $|g^G|$  denote the size of  $g^G$ . The centralizer of  $g$  in  $G$  is denoted by  $C_G(g)$ . Put  $N(G) = \{n \mid \exists g \in G \text{ such that } |g^G| = n\}$ . In 1987 Thompson posed the following conjecture concerning  $N(G)$ .

**Thompson's Conjecture** (see [1], **Question 12.38**). *If  $L$  is a finite simple non-abelian group,  $G$  is a finite group with trivial center, and  $N(G) = N(L)$ , then  $G \simeq L$ .*

Thompson's conjecture was proved for many finite simple groups of Lie type. Denote the alternating group of degree  $n$  by  $Alt_n$  and the symmetric group of degree  $n$  by  $Sym_n$ . Alavi and Daneshkhah proved that the groups  $Alt_n$  with  $n = p$ ,  $n = p + 1$ ,  $n = p + 2$  for prime  $p \geq 5$  are characterized by  $N(G)$  (see [2]). This conjecture has recently been confirmed for  $Alt_{10}$ ,  $Alt_{16}$ ,  $Alt_{22}$  and  $Alt_{26}$  (see [3], [4], [8], [9]). In [5], the author showed that if  $N(G) = N(Alt_n)$  or  $N(G) = N(Sym_n)$  for  $n \geq 5$ , then  $G$  is non-solvable. In [6], we obtained some information about composition factors of a group  $G$  in the case where  $N(G) = N(Alt_n)$  or  $N(G) = N(Sym_n)$  for  $n > 1361$ . It was shown in [7] that Thompson's conjecture is valid for alternating group  $Alt_n$ , where  $n > 1361$ .

Here is our main result.

**Theorem.** *If  $G$  is a finite group with trivial center such that  $N(G) = N(Alt_n)$  or  $N(G) = N(Sym_n)$  with  $n \geq 23$ , then  $G$  has a composition factor  $S$  isomorphic to an alternating group  $Alt_k$ , where  $k \leq n$  and the half-interval  $(k, n]$  contains no primes.*

Theorem and the main result of [7] (see Lemma 1 below) imply immediately the following corollary.

**Corollary.** *Thompson's conjecture is true for an alternating group of degree  $n$  if  $n \geq 5$  and  $n$  or  $n - 1$  is a sum of two primes.*

At this moment it is not known there are even positive integers, which can not be decomposed into a sum of two primes.

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## 2 Notation and preliminary results

**Lemma 1** ([7]). *Let  $G$  be a finite group with trivial center such that  $N(G) = N(\text{Alt}_n)$ , where  $n \geq 27$  and at least one of the numbers  $n$  or  $n - 1$  is a sum of two primes. If  $G$  contains a composition factor  $S \simeq \text{Alt}_{n-\varepsilon}$ , where  $\varepsilon$  is a non-negative integer such that the set  $\{n - \varepsilon, \dots, n\}$  does not contain primes, then  $G \simeq \text{Alt}_n$ .*

**Lemma 2** ([3, Lemma 3]). *If  $G$  and  $H$  are finite groups with trivial centers and  $N(G) = N(H)$ , then  $\pi(G) = \pi(H)$ .*

**Lemma 3** ([6, Lemma 1.4]). *Let  $G$  be a finite group,  $K \trianglelefteq G$ ,  $\overline{G} = G/K$ ,  $x \in G$  and  $\overline{x} = xK \in G/K$ . The following assertions are valid*

- (i)  $|x^K|$  and  $|\overline{x}^{\overline{G}}|$  divide  $|x^G|$ .
- (ii) *If  $L$  and  $M$  are neighboring members of a composition series of  $G$ ,  $L < M$ ,  $S = M/L$ ,  $x \in M$  and  $\tilde{x} = xL$  is an image of  $x$ , then  $|\tilde{x}^S|$  divides  $|x^G|$ .*
- (iii) *If  $y \in G$ ,  $xy = yx$ , and  $(|x|, |y|) = 1$ , then  $C_G(xy) = C_G(x) \cap C_G(y)$ .*
- (iv) *If  $(|x|, |K|) = 1$ , then  $C_{\overline{G}}(\overline{x}) = C_G(x)K/K$ .*

**Lemma 4** ([3, Lemma 4]). *Given a finite group  $G$  with trivial center, if there exists a prime  $p \in \pi(G)$  such that  $p^2$  does not divide  $|x^G|$  for all  $x$  in  $G$ , then the Sylow  $p$ -subgroups of  $G$  are elementary abelian.*

**Lemma 5.** *If  $G = A \times B$ , then  $N(G) = \{ab | a \in N(A), b \in N(B)\}$ .*

*Proof.* The proof is obvious. □

**Lemma 6** ([6, Lemma 1.6]). *Let  $S$  be a non-abelian finite simple group. If  $p \in \pi(S)$ , then there exist  $a \in N(S)$  and  $g \in S$  such that  $|a|_p = |S|_p$ ,  $|g^S| = a$  and  $|\pi(g)| = 1$ .*

**Lemma 7** ([13, Lemma 14]). *Let  $S$  be a non-abelian finite simple group. Any odd element from  $\pi(\text{Out}(S))$  either belongs to  $\pi(P)$  or does not exceed  $m/2$ , where  $m = \max_{p \in \pi(S)}$ .*

**Proposition 1.** *Let  $G$  be a finite group,  $\Omega$  be a set of primes such that, for all  $p, q \in \Omega$  and  $\alpha \in N(G)$ ,  $p$  does not divide  $q - 1$ ,  $p^2$  does not divide  $\alpha$ . Suppose that  $g \in G$  and  $|g| = t \in \Omega$ . If  $\rho = \pi(|g^G|) \cap \Omega \neq \emptyset$ , then  $G$  has a nonabelian composition factor  $S$  and an element  $a \in S$  such that  $|a| = t$  and  $\rho \subseteq \pi(|a^S|)$ .*

*Proof.* Note that, as follows from Lemma 4, a Sylow  $t$ -subgroup of  $G$  is elementary abelian and, consequently,  $|g^G|$  is not a multiple of  $t$ . Let  $K$  be a maximal normal subgroup of  $G$  such that the image  $\overline{g}$  of  $g$  in the group  $\overline{G} = G/K$  is nontrivial and  $\rho \subseteq \pi(|\overline{g}^{\overline{G}}|)$ . Let  $R$  be a minimal normal subgroup of  $\overline{G}$ . It can be represented as the direct product  $R = R_1 \times R_2 \times \dots \times R_l$  of  $l$  isomorphic simple groups. Order of  $R$  is a multiple of some  $r \in \rho \cap \{t\}$ . It follows from Lemmas 5 and 6 that if  $l > 1$ , then there exists  $h \in R$  such that  $|h^G|_r > r$ . It follows from Lemma 3 that there exists  $x \in G$  such that  $|x^G|_r > r$ ; a contradiction. We have  $l = 1$  and hence  $R$  is a simple group. The rest of the proof is divided into two lemmas.

**Lemma 8.** *If  $|\overline{g}^R|$  is a multiple of some  $r \in \rho$ , then Proposition 1 is true.*

*Proof.* Assume that  $|\overline{g}^R|$  is a multiple of some  $r \in \rho$ . It follows from Lemma 7 that  $\overline{g}$  acts on  $R$  as an interior automorphism. Consequently, there exists  $z \in R$  such that  $h^z = h^{\overline{g}}$  for all  $h \in R$ .

It follows from Lemma 6 and the fact that  $r^2$  does not divide  $\alpha$  for any  $r \in \Omega$  and  $\alpha \in N(G)$  that  $k^2$  does not divide  $|R|$  for any  $k \in \Omega$ . Suppose that there exists  $r_1 \in (\pi(|z^{\bar{G}}|) \cap \Omega) \setminus \pi(|z^R|)$ . Since some Sylow  $r_1$ -subgroup of  $R$  centralizes  $z$ , it follows from Frattini argument that some Sylow  $r_1$ -subgroup  $H$  of  $\bar{G}$  normalizes  $\langle z \rangle$ . Since  $|Aut(\langle z \rangle)| = t(t-1)$ , the subgroup  $H$  centralizes  $\langle z \rangle$  and  $r_1 \notin \pi(|z^{\bar{G}}|)$ , a contradiction. Thus,  $\pi(|z^{\bar{G}}|) \cap \rho = \pi(|z^R|) \cap \rho$ .

Assume that there exists  $r_2 \in \rho \setminus \pi(|z^R|)$ . Let us show that  $\bar{G}$  contains an element  $h$  of order  $r_2$  such that  $t \in \pi(|h^{\bar{G}}|)$  and  $R \leq C_{\bar{G}}(h)$ . Let  $\bar{K} < \bar{G}$  be a maximal normal subgroup such that  $|\bar{g}^{\bar{G}/\bar{K}}|$  is a multiple of  $r_2$ , where  $\bar{g} \in \bar{G} = \bar{G}/\bar{K}$  is image of  $g$ . Let  $\tilde{R} < \tilde{G}$  be a minimal normal subgroup. It is clear that  $|\tilde{R}|$  is a multiple of  $r_2$  or  $t$ . As above, it can be shown that  $\tilde{R}$  is a simple group. Since  $\tilde{R}$  contains no outer automorphisms of order  $t$  and  $r_2$ , we obtain that any element of orders  $t$  or  $r_2$  induces an inner automorphism of  $\tilde{R}$ . Since  $\bar{K}$  is maximal, we have that  $|(\bar{g}\tilde{R}/\tilde{R})^{\tilde{G}/\tilde{R}}|$  is not a multiple of  $r_2$ . Therefore  $C_{\tilde{G}/\tilde{R}}(\bar{g}\tilde{R}/\tilde{R})$  contains some Sylow  $r_2$ -subgroup  $T$  of  $\tilde{G}/\tilde{R}$ . Let  $H$  be the full preimage of  $\langle \bar{g}\tilde{R}/\tilde{R}, T \rangle$ . Hence,  $H \simeq (\tilde{R} \times T) \rtimes \langle \bar{g} \rangle$ . Since  $H$  contains some Sylow  $r_2$ -subgroup and an element  $\bar{g}$ , we have  $r_2 \in \pi(|\bar{g}^H|)$ . Therefore  $|\bar{g}^{\tilde{R}}|$  is a multiple of  $r_2$ . As above, it can be shown that there exists an element  $\tilde{h} \in \tilde{R}$  such that  $|\tilde{h}^{\tilde{R}}|$  is a multiple of  $t$ . Since  $\bar{K} > R$  and any  $r_2$ -element acts on  $R$  as an inner automorphism, the element  $\tilde{h}$  have some preimage  $h \in \bar{G}$  such that  $h$  acts trivially on  $R$ . Since  $|\bar{h}^{\bar{G}}|$  divides  $|h^{\bar{G}}|$ , we have  $t \in \pi(|h^{\bar{G}}|)$ .

It follows from Lemma 6, that there exists  $u \in R$  such that  $|u| \neq r_2$  and  $t \in \pi(|u^R|)$ . It follows from Lemma 3 that  $t^2$  divides  $|\overline{uh}^{\bar{G}}|$ , a contradiction. Thus,  $\rho \in |z^R|$ . Therefore  $z$  is the desired element and  $R = S$ .  $\square$

**Lemma 9.**  $\pi(|\bar{g}^R|) \cap \rho \neq \emptyset$ .

*Proof.* Assume that  $\pi(|\bar{g}^R|) \cap \rho = \emptyset$ . Since  $K$  is a maximal subgroup, there exists  $r \in \rho \setminus \pi(|\bar{g}^R/R|)^{\bar{G}/R}$ . Since  $\pi(|\bar{g}^R|) \cap \rho = \emptyset$ , we see that  $C_R(\bar{g})$  contains some Sylow  $r$ -subgroup  $H$  of  $R$ . Let  $N = N_{\bar{G}}(H)$ ,  $T = N_R(H)$ ,  $\bar{N} = N/H$ ,  $\bar{T} = T/H$ ,  $\bar{g}' = \bar{g}H/H$ . From Frattini argument it follows that  $N/T \simeq \bar{G}/R$ , in particular,  $r \notin \pi(|(\bar{g}'T/T)^{N/T}|)$ . Since  $N$  contains some Sylow  $r$ -subgroup of  $\bar{G}$  and the element  $\bar{g}$ , we obtain  $r \in \pi(|\bar{g}^{N\bar{G}(H)}|)$ . Since  $H \leq C_N(\bar{g})$  and  $(|\bar{g}|, r) = 1$ , we have  $r \in \pi(|\bar{g}'^{\bar{N}}|)$ . Let  $F \in Syl_t(\bar{T})$  be such that  $\bar{g}' \in C_{\bar{N}}(F)$ ,  $L = N_{\bar{T}}(F)$ ,  $B = N_{\bar{N}}(F)$ ,  $\tilde{g}' = \bar{g}'F/F$ ,  $\tilde{L} = L/F$ ,  $\tilde{B} = B/F$ . From Frattini argument it follows that  $B$  contains some Sylow  $r$ -subgroup  $A$  of  $\bar{N}$ . Since  $|R|_t \leq t$ , we have  $|F| \leq t$ , in particular,  $A \in C_{\bar{N}}(F)$ . Therefore  $r \in \pi(|\tilde{g}'^{\tilde{B}}|)$ . Since  $r, t \notin \pi(|\tilde{L}|)$ , it follows that  $r \in \pi(|(\tilde{g}'\tilde{L})^{\tilde{B}/\tilde{L}}|)$ ,  $\tilde{L}/\tilde{B} \simeq \bar{G}/R$ , a contradiction.  $\square$

The statement of the proposition follows from Lemmas 8 and 9.  $\square$

**Lemma 10.** Let  $G, \Omega$  and  $g$  be as in Proposition 1. If there exists  $r \in \pi(|g^G|) \cap \Omega$ , then  $G$  contains an element  $h$  of order  $r$  such that  $t \in \pi(|h^G|)$ .

*Proof.* By Lemma 1,  $G$  has a composition factor  $S$  such that  $\bar{g} \in S$ ,  $|\bar{g}| = t$  and  $|\bar{g}^S|$  is a multiple of  $r$ . It follows from Lemmas 6 and 3 that Sylow  $t$ - and  $r$ -subgroups of  $S$  are cyclic groups of simple orders. Let  $\bar{h} \in S$  and  $|\bar{h}| = r$ . Assume that  $|\bar{h}^S|$  is not a multiple of  $t$ . Then, there exists an element  $x \in S$ ,  $|x| = t$ , such that  $\langle x \rangle < C_S(\bar{h})$ . The subgroup  $\langle x \rangle$  is a Sylow  $t$ -subgroup of  $S$ . Consequently, there exists  $y \in S$  such that  $(\langle x \rangle)^y = \langle \bar{g} \rangle$  and, hence,  $\bar{h}^y \in C_S(\bar{g})$ , a contradiction. Thus,  $t \in \pi(|\bar{h}^S|)$ . Consequently,  $G$  contains an element  $h$  of order  $r$  such that  $|h^G|$  is a multiple of  $t$ .  $\square$

From now let us fix  $V_n \in \{Alt_n, Sym_n\}$ , for  $n \geq 5$ , and a finite group  $G$  such that  $N(G) = N(V_n)$ . Let  $\Omega = \{t | t \text{ is a prime, } n/2 < t \leq n\}$  be an ordered set, and  $p$  be the largest number of  $\Omega$ .

**Lemma 11** ([6, Lemma 2.3]). *Suppose that  $t \in \Omega$ ,  $\alpha \in N(G)$  and  $\alpha$  is not a multiple of  $t$ . Then  $\alpha$  is  $|V_n|/t|C|$  or  $|V_n|/|V_{t+i}||B|$ , where  $C = C_{V_{n-t}}(g)$  for some  $g \in V_{n-t}$ ,  $t+i \leq n$ , and  $B = C_{V_{n-t-i}}(h)$  for some  $h \in V_{n-t-i}$ .*

Put  $\Phi_t = \{\alpha \in N(G) \mid \alpha = |V_n|/(t|C|), C = C_{V_{n-t}}(g) \text{ for some } g \in V_{n-t}\}$  and  $\Psi_t = \{\alpha \in N(G) \mid \alpha = |V_n|/(|V_{t+i}||B|) \text{ for some } i \geq 0 \text{ and } t+i < n-1 \text{ where } B = C_{V_{n-t-i}}(g) \text{ for some } g \in V_{n-t-i} \text{ such that } g \text{ moves } n-t-i \text{ points}\}$ . Observe that the definitions of the sets  $\Phi$  and  $\Psi$  do not assume that  $t$  is a prime.

**Lemma 12** ([6, Lemma 2.4]). *Let  $t_i \in \Omega, 1 \leq i < |\Omega|$ . The set  $\Psi_{t_i} \setminus \Psi_{t_{i+1}}$  is empty iff  $n - t_i = 2$  and  $V = Alt$ .*

Given a finite set of positive integers  $\Theta$ , we define a directed graph  $\Gamma(\Theta)$  with the vertex set  $\Theta$  and with edges  $\overrightarrow{ab}$  whenever  $a$  divides  $b$ . Denote by  $h(\Theta)$  the maximal length of a path in  $\Gamma(\Theta)$ .

**Lemma 13** ([5, Lemma 8]). *The following claims hold:*

1. *If  $n - t = 2$  then  $h(\Psi_p) \leq 1$ ;*
2. *If  $n - t = 3$  then  $h(\Psi_p) \leq 2$ ;*
3. *If  $n - t = 4$  then  $h(\Psi_p) \leq 3$ ;*
4. *If  $n - t = 5$  then  $h(\Psi_p) \leq 5$ ;*
5. *If  $n - t = 6$  then  $h(\Psi_p) \leq 6$ ;*
6. *If  $n - t = 7$  then  $h(\Psi_p) \leq 8$ ;*
7. *If  $n - t = 8$  then  $h(\Psi_p) \leq 11$ ;*
8. *If  $n - t = 9$  then  $h(\Psi_p) \leq 14$ ;*
9. *If  $n - t = 10$  then  $h(\Psi_p) \leq 18$ ;*
10. *If  $n - t = 11$  then  $h(\Psi_p) \leq 21$ ;*
11. *If  $n - t = 12$  then  $h(\Psi_p) \leq 26$ ;*
12. *If  $n - t = 13$  then  $h(\Psi_p) \leq 30$ ;*
13. *If  $n - t = 18$  then  $h(\Psi_p) \leq 69$ .*

**Lemma 14.** *The following claims hold:*

1. *If  $23 \leq n \leq 26$  then  $h(\Psi_p) \leq 2$ ;*
2. *If  $31 \leq n \leq 36$  then  $h(\Psi_p) \leq 2$ ;*

3. If  $113 \leq n \leq 124$  then  $h(\Psi_p) \leq 11$ ;
4. If  $139 \leq n \leq 148$  then  $h(\Psi_p) \leq 9$ ;
5. If  $199 \leq n \leq 210$  then  $h(\Psi_p) \leq 12$ ;
6. If  $211 \leq n \leq 222$  then  $h(\Psi_p) \leq 12$ ;
7. If  $317 \leq n \leq 336$  then  $h(\Psi_p) \leq 17$ ;
8. If  $523 \leq n \leq 540$  then  $h(\Psi_p) \leq 26$ ;
9. If  $887 \leq n \leq 905$  then  $h(\Psi_p) \leq 35$ ;
10. If  $1129 \leq n \leq 1150$  then  $h(\Psi_p) \leq 39$ ;
11. If  $1327 \leq n \leq 1360$  then  $h(\Psi_p) \leq 58$ ;

*Proof.* Using [10] we obtain the required bounds for  $h(\Psi_p)$ . □

Let  $\pi$  be some set of primes. A finite group is said to have property  $D_\pi$  if it contains a Hall  $\pi$ -subgroup and all its Hall  $\pi$ -subgroups are conjugate. For brevity, we will write  $G \in D_\pi$  if a group  $G$  has the property  $D_\pi$ .

**Lemma 15** ([11, Corollary 6.7]). *Suppose that  $G$  is a finite group and  $\pi$  is some set of primes. Then  $G \in D_\pi$  if and only if each composition factor of  $G$  has property  $D_\pi$ .*

**Lemma 16** ([12]). *Suppose that  $G$  is a finite group and  $\pi$  is some set of primes. If  $G$  has a nilpotent Hall  $\pi$ -subgroup, then  $G \in D_\pi$ .*

### 3 Proof of Theorem

Take a finite group  $G$  with  $N(G) = N(V_n)$ , where  $5 \geq n \geq 1361$  and  $Z(G) = 1$ . Put  $\Omega = \{t \mid n/2 < t \leq n, t \text{ is a prime}\}$  and denote the largest number of  $\Omega$  by  $p$ . Lemma 2 implies that  $\pi(G) \supseteq \pi(V_n)$ , in particular,  $\Omega \subseteq \pi(G)$ . In view of the main result of [2], we assume that for  $V_n = \text{Alt}_n$  the numbers  $n$  and  $n - 1$  are not prime.

**Proposition 2.** *There exists an element  $g \in G$  such that  $|g| \in \Omega$  and  $|g^G| \in \Phi_{|g|}$ .*

*Proof.* Assume that there is no element  $g \in G$  such that  $|g| \in \Omega$  and  $|g^G| \in \Phi_{|g|}$ .

**Lemma 17.** *If  $|g| \in \Omega$ , then  $|g^G| \in \Psi_p$ .*

*Proof.* Assume that  $|g^G| \in \Psi_{|g|} \setminus \Psi_p$ . Then  $|g^G|$  is a multiple of  $p$ . Consequently,  $|g| \neq p$ . By Lemma 10,  $G$  contains an element  $h$  of order  $p$  such that  $|g| \in \pi(|h^G|)$  and, consequently,  $|h^G| \notin \Psi_p$ . It follows from Lemma 11 that  $|h^G| \in \Phi_p$ ; a contradiction. □

**Lemma 18.**  $G \in D_\Omega$ .

*Proof.* It follows from Lemma 15 that it is sufficient to verify the property  $D_\Omega$  for every composition factor of  $G$ . Let  $S$  be a nonabelian composition factor of  $G$  such that  $|\pi(S) \cap \Omega| \geq 2$  and let  $r$  and  $t$  be two different elements from  $\pi(S) \cap \Omega$ . By Lemma 3, there are no multiples of  $r^2$  or  $t^2$  in  $N(S)$ . By Lemma 6, a Sylow  $a$ -subgroup has order  $a$  for any  $a \in \{r, t\}$ . By Lemmas 17 and 3,  $S$  contains a Hall  $\{r, t\}$ -subgroup  $H$  of order  $rt$ . By the definition of the numbers  $r$  and  $t$ , the group  $H$  is cyclic. It follows from Lemma 16, that  $S \in D_{\{t, r\}}$ . Let  $l \in \pi(S) \cap \Omega \setminus \{t, r\}$ ,  $g \in S$ ,  $|g| = l$ . Since  $|g^S|$  is not multiple of  $t$  or  $r$ , we have  $H^x < C_S(g)$  for some  $x \in S$ . Consequently,  $S$  contains a cyclic Hall  $\{t, r, l\}$ -subgroup. Using Lemma 16, we obtain that  $S \in D_{\{t, r, l\}}$ . Repeating this procedure  $|\pi(S) \cap \Omega|$  times, we obtain that  $S \in D_\Omega$ .  $\square$

**Lemma 19.** *Hall  $\Omega$ -subgroup of  $G$  is abelian.*

*Proof.* As follows from Lemma 4, Sylow  $t$ -subgroup of  $G$  is abelian for any  $t \in \Omega$ . Assume that a Hall  $\Omega$ -subgroup of  $G$  is nonabelian. Then  $G$  contains a nonabelian Hall  $\{r, t\}$ -subgroup  $H$  for some  $r, t \in \Omega$ . Let  $R < G$  be a maximal normal subgroup such that the image  $\overline{H}$  of  $H$  in the group  $\overline{G} = G/R$  is nonabelian. The group  $\overline{H}$  contains a normal  $l$ -subgroup  $T$ , for some  $l \in \{r, t\}$ . Note that  $g \in \overline{G}$ ,  $|g| \in \{r, t\} \setminus \{l\}$ , acts nontrivially on  $T$ . We have  $T = C_T(g) \times [T, g]$ , where  $\langle g, [T, g] \rangle$  is a Frobenius group. Since  $l - 1$  is not a multiple of  $|g|$ , we obtain that  $|[T, g]| > l$  and  $T$  is a acyclic group. By the definitions of the groups  $R$  and  $T$ , we obtain that  $T$  lies in some minimal normal subgroup  $K$  of the group  $G$ . If  $K$  is solvable, then  $K = T$  is an elementary abelian group and, consequently, the subgroup  $K \cap C_K(g)$  is a Sylow  $l$ -subgroup of  $C_K(g)$ . As follows from 3,  $G$  contains a preimage  $h$  of  $g$  such that  $|h^G|$  is a multiple of  $|[T, g]|$ , a contradiction. Therefore,  $K = S_1 \times S_2 \times \dots \times S_m$ , where  $S_i$  is a nonabelian simple group for  $1 \leq i \leq m$ . It can be shown as in Lemma 1 that  $m = 1$ . As noted in Lemma 18, Hall  $\{r, t\}$ -subgroup of any composition factor is cyclic. We obtain a contradiction with the fact that  $K$  contains an acyclic  $l$ -subgroup  $T$ .  $\square$

**Lemma 20.** *If  $T$  be a Hall  $\Omega$ -subgroup of  $G$  and  $\Upsilon = \{|g^G|, g \in T\}$ , then  $|\Omega| \leq h(\Upsilon)$ .*

*Proof.* Let  $g_1 \in T$  and  $|g_1| = t_1 \in \Omega$ . By Lemma 12,  $G$  contains an element  $r_1 \in G$  such that  $|r_1^G| \in \Psi_{t_1} \setminus \Psi_{t_2}$ , where  $t_2$  is the smallest number of  $\Omega \setminus \{t_1\}$ . Since  $G \in D_\Omega$  (see Lemma 18), we can assume that  $r_1 \in C_G(g_1)$  and a Hall  $\Omega$ -subgroup of  $C_G(r_1)$  lies in  $T$ . Consequently, there exists  $g_2 \in T$  such that  $|g_2| = t_2$  and  $C_G(g_1) \neq C_G(g_2)$ . By Lemma 19, the group  $T$  is abelian. Thus,  $|(g_1 g_2)^G| > |g_1^G|$ . Note that  $|g_1^G|$  divides  $|(g_1 g_2)^G|$ . Repeating this procedure  $|\Omega|$  times, we obtain the set  $\Sigma = \{g_1, g_1 g_2, g_1 g_2 g_3, \dots, g_1 g_2 \dots g_{|\Omega|}\}$  such that  $|g_1^G|$  divides  $|(g_1 g_2)^G| \dots |g_1 g_2 \dots g_{|\Omega|}|$ . In particular,  $h(\Upsilon) \geq |\Omega|$ .  $\square$

**Lemma 21.**  $n \notin \{27, 28, 125, 126\}$

*Proof.* Assume that  $n \in \{27, 28, 125, 126\}$ . Let  $t = 3$  if  $n \in \{27, 28\}$  and  $t = 5$  if  $n \in \{125, 126\}$ ,  $T \in \text{Syl}_G(t)$ ,  $g \in C_G(T)$ . We have  $|g^G| = |V_n|/|V_n|_t$ , in particular,  $|g^G|$  is a minimal and maximal element in the set  $N(G) \setminus \{1\}$ . Let  $K \trianglelefteq G$  be a minimal normal subgroup such that  $5 \in \pi(K)$ . Assume that there exists  $r \in \Omega \cap \pi(G/K)$ . Let  $h \in G$ ,  $|h| = r$ . Without loss of generality, we can assume that  $h \in N_G(Z(T \cap K))$  and  $T \cap C_G(h) \in \text{Syl}_{N_G(h)}(t)$ . Since any multiple  $a \in N(G)$  of  $|h^G|$  is not a multiple of  $|g^G|$  for any  $g' \in Z(T)$ , the element  $h$  acts without fixed points on  $Z(T \cap K)$ . Therefore  $|h^G|_t \geq t^x$ , where  $x$  is such that  $|h|$  divides  $p^x - 1$ . But  $p^x > |\alpha|_p$  for any  $\alpha \in \text{Psi}_p$ ; a contradiction. Let  $R \triangleleft K$  be a maximal normal subgroup,

$\Upsilon = \{17, 23\}$  if  $n \in \{27, 28\}$  and  $\Upsilon = \{113, 109\}$  if  $n \in \{125, 126\}$ . Using Frattini argument we get that the set  $\pi|R| \cap \Upsilon$  is empty. Hence,  $\Upsilon \subseteq \pi(K/R)$ . Since  $R$  is maximal,  $K/R$  is simple. It follows from [14], that  $K/R \in \{Alt_{23}, Alt_{24}, \dots, Alt_{28}, Fi_{23}, Alt_{113}, \dots, Alt_{126}, U_7(4)\}$ . Since  $G \in D_\Omega$ , we have  $K/R \in D_\Omega$ . Let  $O$  be a Hall  $\Upsilon$ -subgroup of  $K/R$ . It follows from Lemma 19, that  $O$  is abelian, a contradiction.  $\square$

It follows from Lemma 21, that  $n \notin \{27, 28, 125, 126\}$ . We obtain From Lemmas 13 and 14, that  $|\Omega| > h(\Psi_p)$ , a contradiction with Lemma 20. This completes the proof of Proposition 2.  $\square$

It follows from Proposition 2, that there exists an element  $g \in G$  such that  $|g| \in \Omega$  and  $|g^G| \in \Phi_{|g|}$ . It follows from Lemma 1 that there exists a composition factor  $S$  of  $G$  and  $\bar{g} \in S$  such that  $|\bar{g}| = |g|$ ,  $\pi(|\bar{g}^S|) \cap \Omega = \Omega \setminus \{|g|\}$  and  $|S|$  is not a multiple of  $r^2$  for any  $r \in \Omega$ .

**Lemma 22.** *If  $n \geq 23$ , then  $S$  is not isomorphic to a finite simple group of Lie type.*

*Proof.* As noted above,  $\Omega \subset \pi(S)$ . Let  $\Lambda(q^m)$  be an exceptional group of Lie type over field of order  $q^m$ , where  $q < 663$  is a prime,  $m \geq 1327$ . Using [10] we can obtain that  $\pi(\Lambda(q^m))$  either contains a number greater than 1327 or does not contains  $\Omega(r)$ , where  $\Omega(r) = \{t \mid r/2 < t < r, t \text{ is a prime}\}$  for  $r$  greater or equal than the maximal number of  $\pi(\Lambda(q^m))$ . Since  $\pi(\Lambda(q^m))$  contains a primitive divisor of  $q^m - 1$ , we obtain that if  $m > 1327$ , then  $\pi(\Lambda(q^m))$  contains a number greater than 1327.  $|\Lambda(q^m)|$  is a multiple of  $q^2$ . Therefore  $S$  is not isomorphic to an exceptional group of Lie type. Let  $L = \Lambda_k(q^m)$  be a finite simple classical group of Lie type of Lie rank  $k$  over the field  $q^m$ . The set  $\pi(L)$  contains a primitive divisor of  $q^{mk} - 1$  or a primitive divisors of  $q^{2mk} - 1$ . Hence, if  $mk > 1327$ , then  $\pi(L)$  contains a number greater than 1327. If  $k > 1$ , then  $|L|$  is a multiple of  $q^2$ . Using [10] we get that  $\pi(L)$  contains a number greater than 1327 or does not contain  $\Omega(r)$  for  $r$  greater or equal than the maximal number of  $\pi(\Lambda_k(q^m))$ .  $\square$

**Lemma 23.**  *$S$  not isomorphic to a finite simple sporadic group.*

*Proof.* The proof is obvious.  $\square$

It follows from Lemmas 22 and 23, that  $S \simeq Alt_m$ . Since  $p \in \pi(S)$ , we obtain the required equality  $m \geq p$ . Thus, Theorem is proved. Corollary follows from Theorem and Lemma 1.

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